Semilinear ordinary differential equation coupled with distributed order fractional differential equation

Teodor M. Atanackovic[*] Ljubica Oparnica [†] Stevan Pilipović [‡]

Abstract

System $D^2y(t) + z(t) = f(t,y)$, $\int_0^2 \phi_1(\gamma)D^\gamma y(t)d\gamma = \int_0^2 \phi_2(\gamma)D^\gamma z(t)d\gamma$, $t > 0$, where $D^\gamma, \gamma \in [0,2]$ are fractional derivatives, is investigated and the existence of the solution in a mild and classical sense is proven. Such a system arises as a distributed derivative model of viscoelastic body and in the system identification theory. Also, the condition for existence and uniqueness of a solution to a general linear fractional differential equation $\sum_{i=1}^k a_i D^{\gamma_i} z(t) = g(t)$, $\gamma_i \in \mathbb{R}$, $i = 1, ..., k$ in $\mathscr{S}'_+$ is given.

Keywords: Fractional differentiation, Tempered distributions, Schauder fixed point theorem.

AMS Subject Classification Subject classification: 26A33; 34G20; 47H10

1 Introduction

In this paper, we solve and analyze solutions to a nonlinear system motivated by a mechanical model

$$(CP) \quad \begin{cases} D^2y(t) + z(t) = f(t,y(t)), & t > 0, \\[2mm] \int_0^2 \phi_1(\gamma)D^\gamma y(t)d\gamma = \int_0^2 \phi_2(\gamma)D^\gamma z(t)d\gamma, & t > 0, \\[2mm] y(0) = y_0; \qquad y^{(1)}(0) = v_0. \end{cases}$$

Here, y and z, which represent strain and stress respectively, are unknown functions defined for $t > 0$. Given are locally integrable function f representing forcing term, and ϕ_1, ϕ_2 which characterize the material under consideration. Constants y_0 and v_0 are initial displacement and velocity. Operators $D^\gamma, \gamma \in [0,2]$ are operators of usual (for γ integer) or fractional differentiation. System (CP) models various physical processes. For example, $(CP)_1$ represents an equation of motion of a physical pendulum (in this case $f(t,y(t)) = \sin y(t)$), moving in a dissipative media of viscoelastic type; $(CP)_2$ represents a constitutive equation of a material where energy is dissipated. Since the upper bound in integrals in $(CP)_2$ is two, both viscoelastic and viscoinertial effects are included. As experiments show, the presence of integral on the left hand side of $(CP)_2$ indicates that dissipation properties depend on the order of the derivative. The integral on the right hand side of $(CP)_2$ is a consequence of the well known principle of equipresence (cf. [23]).

Fractional operators D^γ and I^γ, $\gamma \in \mathbb{R}$, (see (2) and (1), respectively) are widely used in constitutive equations for modeling viscoelastic bodies. A typical model is the generalized Zener model, where relations between stress and strain involve linear fractional differential operators. We refer to [2], [11] and [8] for details. A viscoelastic model of wave propagation with distributed order

[*]Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 5, 21000 Novi Sad, Serbia, (atanackovic@uns.ns.ac.yu)

[†]Institute of Mathematics, Serbian Academy of Science, Kneza Mihaila 35, 11000 Belgrade, Serbia, (ljubicans@sbb.co.yu)

[‡]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia, (pilipovic@im.ns.ac.yu)

derivatives was analyzed in [9]. In [1] a distributed derivative model of viscoelastic body was proposed, and the restrictions which follow from the Second Law of Thermodynamics are stated. Distributed order derivatives appear in many other branches of physics. For example, Caputo in [10] introduced distributed order fractional derivative to model dielectric induction and diffusion. In the series of papers [12, 13, 19, 20] the fractional diffusion equation, which was obtained from standard diffusion equations by replacing the first-order time derivative with the fractional derivative of order $\beta \in (0,1)$ was analyzed. In [7] and [15] distributed derivative models were used in the context of system identification theory.

Nahušev, [21], considers distributed order derivatives of the form $\int_c^d D^\alpha y d\alpha$, $c, d \in \mathbb{R}$. He used the Laplace transform method to show the existence of a solution to equations of type $\int_c^0 D^\alpha y d\alpha = g$, $c < 0$ and $\int_0^d D^\alpha y d\alpha = g$, $d > 0$ in $L^1[a,b]$, with $g \in L^1$. Recently, Kochubei [18] gave the necessary condition on continuous function ϕ in order that distributed order fractional differential equation of the form $\int_0^d \phi(\gamma) D^\alpha y d\alpha = g$ has a solution. Some examples of mechanical models with distributed order fractional derivative were also studied in our previous works [3], [4] and [6]. Those are all particular cases of system (CP).

In order to solve system (CP) we will use the Laplace transform of tempered distributions supported by the half line and the Schauder fixed point theorem which leads to a solution in an interval $[0, \delta]$, for some $\delta > 0$. Also, we will show existence and uniqueness for a distributed order fractional differential equation $\int_{\operatorname{supp} \phi} \phi(\gamma) D^\gamma z = g$ in $\mathscr{S}'_+$ if ϕ is of the form $\phi = \sum_{i=0}^k a_i \delta(\cdot - \gamma_i)$, $a_i \in \mathbb{R}, a_i \neq 0$, $\gamma_i \in \mathbb{R}$, $i = 0, 1, ..., k$ so that this equation has a form of a linear fractional differential equation $\sum_{i=0}^k a_i D^{\gamma_i} z = g$. We refer to [17] for solutions of linear fractional differential equation in spaces of continuous or locally integrable functions. Also we refer to [16] for solutions within spaces of distributions defined on the hole line obtained by the use of the Fourier transform method.

In Section 2, we will define different types of solutions to (CP) problem: *classical*, *mild* and *non-impact* ones. Further on, in Section 3, we state and prove the existence and uniqueness of a solution to a general linear fractional differential equation in the frame of $\mathscr{S}'_+$, assuming that the corresponding holomorphic function $\sum_{i=0}^k a_i s^{\gamma_i}$, $0 \leq \gamma_i < 2$, $\operatorname{Re} s > 0$, does not have zeros. The existence of solutions to system (CP) is proved in Section 4. In Theorem 4.1, we give sufficient conditions for the existence of *mild*, *non-impact* and *classical* solution to (CP), in the case when ϕ_1 is continuous in $[c, d]$ for some $0 \leq c < d < 2$ and $\phi_2(\cdot) = \sum_{i=0}^k a_i \delta(\cdot - \gamma_i)$, $0 \leq \gamma_i < 2$. Further on, we analyze cases where form of ϕ_2 remains the same, while ϕ_1 changes. In Section 5, we perform our analysis for the case $\phi_1(\gamma) = b^\gamma$ and $\phi_2(\gamma) = a^\gamma$, $b > a$, where $b > a$ is a restriction which follows from the Second Law of Thermodynamics.

1.1 Notation and notions

Let $I \subseteq \mathbb{R}$ be an interval of $\mathbb{R}$. Then, we denote the space of locally integrable functions in I by $L^1_{\text{loc}}(I)$, the space of k-times continuously differentiable functions y by $C^k(I)$, and the space of functions $y \in C^{k-1}(I)$, such that $y^{(k)} \in L^1_{\text{loc}}(I)$ by $AC^k(I)$.

The space of functions $y \in L^1_{\text{loc}}(\mathbb{R})$, with the property $y(t) = 0$, $t < 0$ we will denote by $L^1_{\text{loc}+}(\mathbb{R})$. Further, $\mathscr{D}(\mathbb{R})$ and $\mathscr{S}(\mathbb{R})$ are the spaces of compactly supported smooth functions and of rapidly decreasing smooth functions in $\mathbb{R}$, respectively. Their duals $\mathscr{D}'(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ are the spaces of Schwartz's and tempered distributions, respectively. $\mathscr{D}'_+(\mathbb{R})$ and $\mathscr{S}'_+(\mathbb{R})$ denote their subspaces consisting of distributions supported by $[0, \infty)$. Henceforth, we drop $\mathbb{R}$ in the notation of $\mathscr{D}'_+(\mathbb{R})$ and $\mathscr{S}'_+(\mathbb{R})$. We will also use compactly supported distributions, elements of $\mathcal{E}'(\Omega)$, where Ω is an open interval of $\mathbb{R}$. Any element of $\mathcal{E}'(\Omega)$ can be extended to $\mathbb{R}$ as an element of $\mathscr{D}'(\mathbb{R})$ in the usual way. Let $J \subseteq I$ be a subinterval of I and let y be a function or a distribution defined in I. We will use the same notation y for its restriction in J. Thus $y|_J$ will be denoted by y again. In this sense, we will write $f \in \mathscr{S}'_+ \cap C^p([0,a])$, which means that $f \in \mathscr{S}'_+$ and $f|_{[0,a]} \in C^p([0,a])$.

The Riemann-Liouville operator of fractional differentiation or integration is denoted by D^γ, $\gamma \in \mathbb{R}$ defined as follows. Let $y \in L^1_{\text{loc}+}(\mathbb{R})$. The Riemann-Liouville fractional integral of order $\gamma > 0$ is

defined by

$$I^\gamma y(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} y(\tau) d\tau, \quad t > 0, \tag{1}$$

where Γ is the Euler gamma function. For $\gamma = 0$ one defines $I^0 y := y$. If $y \in L^1_{\text{loc}+}(\mathbb{R})$, it is shown (cf. [22, Theorem 2.6]) that $\lim_{\gamma \to 0} I^\gamma y(t) = y(t)$ almost everywhere in $\mathbb{R}$.

Let $y \in L^1_{\text{loc}+}(\mathbb{R})$ and suppose that for every $a > 0$, $y \in AC^k([0, a])$. The Riemann-Liouville fractional derivative of order $0 \leq \gamma < k$ for some $k \in \mathbb{N}$, is defined by

$$D^\gamma y(t) := \frac{d^k}{dt^k} I^{k-\gamma} y(t), \quad t > 0. \tag{2}$$

If $\gamma \in \mathbb{N}$, then $D^\gamma = \frac{d^\gamma}{dt^\gamma}$.

Riemann-Liouville's fractional operators in the setting of distributions are defined (e.g. [24]) by introducing the family $f_\alpha \in \mathscr{D}'_+$, $\alpha \in \mathbb{R}$

$$f_\alpha(t) := \begin{cases} H(t)\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t \in \mathbb{R}, \alpha > 0, \\ \frac{d^N}{dt^N} f_{\alpha+N}(t), & \alpha \leq 0, \alpha + N > 0, N \in \mathbb{N}, \end{cases}$$

where H is Heaviside's function. Then, $f_\alpha * f_\beta = f_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{R}$ and $f_\alpha *$ is the convolution operator in $\mathscr{D}'_+$. Also, $f_\alpha : \mathscr{S}'_+ \to \mathscr{S}'_+$. The operator $f_\alpha *$ in $\mathscr{D}'_+$ is the operator of fractional differentiation for $\alpha < 0$ and of fractional integration for $\alpha > 0$. It coincides with the operator of derivation for $-\alpha \in \mathbb{N}$ and integration for $\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In particular, $f_0 = \delta$.

If $\alpha > 0$ and $y \in L^1_{\text{loc}+}(\mathbb{R})$ then $I^\alpha y = f_\alpha * y$. If $0 \leq \alpha < k$, $k \in \mathbb{N}$ and for all $a > 0$, $y \in L^1_{\text{loc}+}(\mathbb{R}) \cap AC^k([0, a])$, then $D^\alpha y = f_{-\alpha} * y$. We will use notion $D^\alpha y$ also when y is distribution and will denote $D^{-\gamma} y = I^\gamma y$, $\gamma > 0$. Note that $D^\gamma I^\gamma y = y$ for $y \in L^1_{\text{loc}+}(\mathbb{R})$ and $I^\gamma D^\gamma y = y$, $\gamma > 0$ if y is considered as an element in $\mathscr{D}'_+$.

Let $0 \leq c < d$, $\phi \in C([c, d])$ and $y \in L^1_{\text{loc}+}(\mathbb{R}) \cap AC^2([0, a])$. Then the distributed order fractional derivative in $[c, d]$ is given by

$$\int_c^d \phi(\gamma) D^\gamma y(t) d\gamma. \tag{3}$$

If $\phi : [c, d] \to \mathbb{R}_+$ be a piecewise continuous bounded function and $y \in \mathscr{S}'_+$, then as in [6], one can consider (3) as a Bochner type of integral. We refer to [5] for the properties of (3) within $\mathscr{S}'_+$. Recall the definition from [5]:

Definition 1.1. *Let $\phi \in \mathcal{E}'(\mathbb{R})$ and $y \in \mathscr{S}'_+$. Then $\int\limits_{\text{supp}\,\phi} \phi(\gamma) D^\gamma y\, d\gamma$ is defined as an element of $\mathscr{S}'_+$ by*

$$\langle \int\limits_{\text{supp}\,\phi} \phi(\gamma) D^\gamma y(t)\, d\gamma, \varphi(t) \rangle := \langle \phi(\gamma), \langle D^\gamma y(t), \varphi(t) \rangle \rangle, \quad \varphi \in \mathscr{S}(\mathbb{R}).$$

When $\text{supp}\,\phi \subset [a, b]$ we write $\int_a^b \phi(\gamma) D^\gamma y\, d\gamma$ instead of $\int\limits_{\text{supp}\,\phi} \phi(\gamma) D^\gamma y\, d\gamma$. It is shown in [5] that $\gamma \mapsto \langle D^\gamma y, \phi \rangle : \mathbb{R} \to \mathbb{R}$ is smooth and that the mapping $(\alpha, y) \mapsto D^\alpha y$ is continuous from $\mathbb{R} \times \mathscr{S}'_+$ to $\mathscr{S}'_+$. Also, $y \mapsto \int\limits_{\text{supp}\,\phi} \phi(\gamma) D^\gamma y\, d\gamma$ is a linear and continuous mapping from $\mathscr{S}'_+$ to $\mathscr{S}'_+$. For

$$\phi_2(\cdot) = \sum_{i=0}^k a_i \delta(\cdot - \gamma_i), \quad \gamma_i \in \mathbb{R}\, i \in \{0, 1, ..., k\} \tag{4}$$

distributed order fractional derivatives becomes a sum of finite number derivatives of fractional order $\sum_{i=0}^k a_i D^{\gamma_i} y$.

Distributed order fractional differential equation is given by

$$\int_{\operatorname{supp}\phi} \phi(\gamma)D^\gamma y \, d\gamma = g, \tag{5}$$

while for ϕ of the form (4), it becomes linear fractional differential equation.

To deal with fractional differential equations, it is convenient to use the Laplace transform. Recall, if $y \in \mathscr{S}'_+$, then its Laplace transform is defined by

$$\widehat{y}(s) = \mathcal{L}y(s) = \langle y(t), \varphi(t)e^{-st}\rangle, \quad \operatorname{Re} s > 0,$$

where $\varphi \in C^\infty$, $\varphi = 1$ in $(-a, \infty)$ and $\varphi = 0$ in $(-\infty, -2a)$, $a > 0$. Note that $\mathcal{L}y$ is an analytic function for $\operatorname{Re} s > 0$, and that the definition of $\mathcal{L}y$ does not depend on a chosen function φ with given properties. We will use the identities given in the next proposition (see [5]).

Proposition 1.2. *Let $\phi \in \mathcal{E}'(\mathbb{R})$ and $y \in \mathscr{S}'_+$. Then:*

a) $\mathcal{L}(f_\alpha * y)(s) = \frac{1}{s^\alpha}\widehat{y}(s)$, $\operatorname{Re} s > 0, \alpha \in \mathbb{R}$.

b) $\mathcal{L}(\int_{\operatorname{supp}\phi} \phi(\gamma)D^\gamma y \, d\gamma)(s) = \widehat{y}(s)\langle\phi(\gamma), s^\gamma\rangle$, $\operatorname{Re} s > 0$.

c) If ϕ is a continuous function in $[c, d]$ and $\phi(\gamma) = 0$, $\gamma \notin [c, d]$, then

$$\mathcal{L}(\int_c^d \phi(\gamma)D^\gamma y \, d\gamma)(s) = \widehat{y}(s)\int_c^d \phi(\gamma)s^\gamma \, d\gamma, \quad \operatorname{Re} s > 0.$$

2 Definitions of solutions to (CP)

Problem (CP) will be analyzed in the context of the following definitions.

Definition 2.1 (Classical solution). *Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and $\phi_1, \phi_2 \in \mathcal{E}'(\mathbb{R})$, $\operatorname{supp}\phi_i \subset [0, 2]$, $i = 0, 1$. Let $y_0, v_0 \in \mathbb{R}$, $\delta > 0$, $y \in \mathscr{S}'_+ \cap C^2([0, \delta])$, $z \in \mathscr{S}'_+ \cap C([0, \delta])$.*
*We say that (y, z) is a **classical solution to** (CP) **in** $[0, \delta]$ if y and z satisfy $\widetilde{(CP)}$, where*

$$\widetilde{(CP)} \quad \begin{cases} D^2 y(t) + z(t) = f(t, y(t)), \quad t \in [0, \delta], \\[2mm] \int_0^2 \phi_1(\gamma)D^\gamma y d\gamma = \int_0^2 \phi_2(\gamma)D^\gamma z d\gamma, \quad \text{in } \mathscr{S}'_+, \\[2mm] y(0) = y_0, \qquad y^{(1)}(0) = v_0. \end{cases}$$

Let $\phi_1, \phi_2 \in \mathcal{E}'(\mathbb{R})$, assume that $y \in \mathscr{S}'_+$ and let $z \in \mathscr{S}'_+$ be a solution to (5) with $\phi = \phi_2$ and $g = \int_0^2 \phi_1(\gamma)D^\gamma y d\gamma$. Then z is a solution to $(CP)_2$ in $\mathscr{S}'_+$. This is provided by

$$\mathcal{L}^{-1}\left(\frac{1}{\langle\phi_2(\gamma), s^\gamma\rangle}\right) \in \mathscr{S}'_+, \tag{6}$$

and then

$$z = \mathcal{L}^{-1}\left(\frac{1}{\langle\phi_2(\gamma), s^\gamma\rangle}\right) * \int_0^2 \phi_1(\gamma)D^\gamma y(\cdot)d\gamma, \quad \text{in } \mathscr{S}'_+. \tag{7}$$

Equation $(CP)_1$ with initial conditions $(CP)_3$, in the setting of tempered distributions, reads

$$f_{-2} * y + z = f(\cdot, y) + y_0\delta' + v_0\delta. \tag{8}$$

We substitute (7) in (8) and apply $D^{-2} = f_2*$. It follows that

$$y = -\mathcal{L}^{-1}\left(\frac{1}{\langle\phi_2(\gamma), s^\gamma\rangle}\right) * \int_0^2 \phi_1(\gamma)D^{\gamma-2}y d\gamma + D^{-2}f(\cdot, y)) + v_0 f_1 + y_0 H.$$

Definition 2.2 (Mild solution). *Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a locally integrable function and $\phi_1, \phi_2 \in \mathcal{E}'(\mathbb{R})$, $\operatorname{supp} \phi_i \subset [0, 2]$, $i = 0, 1$, where ϕ_2 is such that (6) holds. Let $y_0, v_0 \in \mathbb{R}$ and $\delta > 0$. The pair (y, z) is called a* **mild solution** *to* (CP) **in** $[0, \delta]$ *if $y \in \mathscr{S}'_+ \cap C([0, \delta])$, such that*

$$y(t) = -\mathcal{L}^{-1}\left(\frac{1}{\langle \phi_2(\gamma), s^\gamma \rangle}\right)(t) * \int_0^2 \phi_1(\gamma) D^{\gamma-2} y(t) d\gamma$$
$$+ D^{-2} f(t, y(t)) + v_0 t + y_0, \quad t \in [0, \delta], \tag{9}$$

and z is given by (7).

Remark 2.3. a) A mild solution to (CP) in $[0, \delta]$ is a weak solution in $(0, \delta)$. Namely, (9) and (7) imply that for every $\theta \in C_0^\infty((0, \delta))$

$$\langle D^2 y, \theta \rangle = \langle y, D^2 \theta \rangle = \langle D^{-2} z(t) + D^{-2} f(t, y(t)) + v_0 t + y_0, D^2 \theta(t) \rangle$$
$$= \langle z, \theta \rangle + \langle f(t, y(t)), \theta(t) \rangle.$$

b) A classical solution is also a mild solution. If $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and a mild solution exists, then considering that mild solution satisfies $\widetilde{(CP)}_2$, in order for it to be a classical solution, one needs additionally to assume that $y \in C^2([0, \delta])$ and $z \in C([0, \delta])$.

In mechanics, an impact solution is an absolutely continuous function whose first derivative has jumps. Impact solutions are included in the definition of the mild solution. In order to make an additional classification of solutions related to the observable quantity y, we introduce a class of *non-impact solutions*. This class will correspond to solutions in which the first derivative y' (velocity) is continuous.

Definition 2.4 (Non-impact solution). *Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a locally integrable function, $\phi_1, \phi_2 \in \mathcal{E}'(\mathbb{R})$, $\operatorname{supp} \phi_i \subset [0, 2]$, $i = 0, 1$. Let $y_0, v_0 \in \mathbb{R}$ and $\delta > 0$. The pair (y, z) is called a* **non-impact solution** *to* (CP) **in** $[0, \delta]$ *if $y \in \mathscr{S}'_+ \cap AC^2([0, \delta])$ and $z \in \mathscr{S}'_+ \cap L^1([0, \delta])$, such that*

$$D^2 y(t) + z(t) = f(t, y(t)), \quad \text{almost everywhere} \quad t \in [0, \delta],$$
$$\int_0^2 \phi_1(\gamma) D^\gamma y(t) d\gamma = \int_0^2 \phi_2(\gamma) D^\gamma z(t) d\gamma, \quad \text{in } \mathscr{S}'_+$$

and

$$y(0) = y_0, \quad Dy(0) = v_0.$$

It is clear that condition (6) plays an important role in definitions of the solutions, and that satisfaction of (6) depends on ϕ_2. However, in the next section, we will see that when ϕ_2 is of the form (4) with additional condition (A_0) satisfied, then (6) is fulfilled. As we will see in Section 4, (6) and certain conditions on f assure the existence of mild, non-impact, and classical solutions.

3 Linear fractional differential equation in $\mathscr{S}'_+$

In this section, we prove the existence and the uniqueness of a fundamental solution to a general linear fractional differential equation in $\mathscr{S}'_+$.

Let $g \in \mathscr{S}'_+$. Let $a_i \neq 0$ and γ_i be arbitrary real numbers, such that $\gamma_i > \gamma_{i+1}$, $i \in \{0, 1..., k\}$, and consider

$$\sum_{i=0}^k a_i D^{\gamma_i} z = g, \quad \text{in } \mathscr{S}'_+. \tag{10}$$

Theorem 3.1. *Equation (10) has a unique solution $z \in \mathscr{S}'_+$, if and only if*

$$(A_0) \qquad \sum_{i=0}^k a_i s^{\gamma_i} \neq 0, \quad s \in \mathbb{C}_+ = \{s \in \mathbb{C} \,;\, \operatorname{Re} s > 0\}.$$

Proof. First note

$$\sum_{i=0}^{k} a_i D^{\gamma_i} z = \sum_{i=0}^{k} a_i (f_{-\gamma_i} * z) = (\sum_{i=0}^{k} a_i f_{-\gamma_i}) * z,$$

i.e. that (10) is equation of a convolution type. For given $\varphi, h \in \mathscr{S}'_+$, the equation $\varphi * z = h$ is uniquely solvable if and only if there exists $a, b \in \mathbb{R}$ and $C > 0$, such that

$$\frac{1}{|\widehat{\varphi}(s)|} \leq C \frac{(1 + |s|)^a}{|\operatorname{Re} s|^b}, \quad s \in \mathbb{C}_+,$$

where $\widehat{\varphi}$ denotes Laplace transform of φ, (see [25, Chap.7, p.50]).

Let us denote

$$F(s) = \widehat{\varphi}(s) = \sum_{i=0}^{k} a_i s^{\gamma_i}, \ s \in \mathbb{C}_+.$$

We will show that $F(s) \neq 0$, $s \in \mathbb{C}_+$ (which is (A_0)) implies that there exist $a \in \mathbb{R}$ and $C > 0$, such that

$$\frac{1}{|F(s)|} \leq C \frac{(1 + |s|)^a}{(\operatorname{Re} s)^a}, \ s \in \mathbb{C}_+. \tag{11}$$

This will imply the theorem.

To begin with, note that function F is analytic in $\mathbb{C} \backslash (-\infty, 0]$. Next observe that $|F(s)|$ behaves like $|a_0 s^{\gamma_0}|$ when $|s|$ is large, and like $|a_k s^{\gamma_k}|$ when $|s|$ is small which implies that there are $r, R \in \mathbb{R}$, $0 < r < R$, such that $F(s)$ has no zeros for $|s| < r$ and $|s| > R$. Further observe that $F(s) \neq 0$ for $\operatorname{Re} s > 0$ implies that $F(s) \neq 0$ for $\operatorname{Re} s < 0$ and therefore the set of zeros of function F is contained in $[-iR, -ir] \cup [ir, iR]$. Thus, it must be a finite set of points iy_j, $j \in \{1, 2...l\}$, since zero set of an analytic function does not have limits in the domain of its analyticity, so in any compact region, the set of zeros can only be a finite set.

Since F is analytic, it follows that in neighborhood of zeros iy_j, $|F(s)|$ behaves as $|s - iy_j|^{m_j}$, $m_j \in \mathbb{N}$, $j \in \{1, 2, .., l\}$. Denote $D_j = \{s \in \mathbb{C}; |s - iy_j| < r_i\}$, for some $r_j > 0$, $j \in \{1, 2...l\}$, $D = \bigcup_{j=1}^{l} D_j$ and $K = \{s \in \mathbb{C}; r \leq |s| \leq R\}$. Choose $a \geq m_j$, $j \in \{1, 2..., l\}$ and $a \geq \gamma_0$. There exist C such that

$$\frac{1}{|F(s)|} \leq C \frac{1}{(\operatorname{Re} s)^a}, \ s \in (\mathbb{C} \backslash K) \cup D.$$

In $K \backslash D$ function F reaches its minimum, denoted by m, so with some new C

$$\frac{1}{|F(s)|} \leq \frac{1}{m} \leq C \frac{(1 + |s|)^a}{(\operatorname{Re} s)^a}, \ s \in K.$$

Putting all together we arrive to (11). □

Remark 3.2. Equation (10) represents a constitutive equation of a viscoelastic body which, coupled with equation $(CP)_1$, represents a mechanical model (CP). We explained in [5] the mechanical aspect of condition (A_0) and here we repeat it briefly. Let $T > 0$ and y be of the form $y(t) = \sum_{i=0}^{k} a_i D^{\gamma_i} z(t)$, $t \in [0, T]$. The dissipation work is given by $A_d = \int_0^T z(t) y^{(1)}(t) dt$ and the dissipation inequality reads $A_d \geq 0$ (see [14]). If (A_0) does not hold, then the dissipation inequality is violated. So, (A_0) is a necessary condition if (CP) represents a mechanical model.

The solution to (10) is of the form $z = l * g$, where l is a fundamental solution to (10), i.e. solution to $\sum_{i=0}^{k} a_i D^{\gamma_i} y = \delta$, with the properties given in next lemma. For the proof see [5].

Lemma 3.3. *Let $\gamma_i \in [0, 2)$ and $\gamma_i > \gamma_{i+1} \geq 0$ for all $i \in \{0, 1..., k\}$. Assume (A_0). Let*

$$l(t) = \begin{cases} \mathcal{L}^{-1} \left(\frac{1}{\sum_{i=0}^{k} a_i s^{\gamma_i}} \right)(t), & t \geq 0 \\ 0, & t < 0. \end{cases} \tag{12}$$

Then:
(i) l is a locally integrable function in $\mathbb{R}$.
(ii) l is absolutely continuous in $\mathbb{R}$, if $\gamma_0 - \gamma_k > 1$.

4   Solvability of the system (CP)

Through this section we consider cases when ϕ_2 is of the following form

$$\phi_2(\cdot) = \sum_{i=0}^{k} a_i \delta(\cdot - \gamma_i), \quad \gamma_i \in [0, 2), \, \gamma_i > \gamma_{i+1}, \, i \in \{0, 1, ..., k\}. \tag{13}$$

The constitutive equation $(CP)_2$ then reads

$$\int_0^2 \phi_1(\gamma) D^\gamma y(t) d\gamma = \sum_{i=0}^{k} a_i D^{\gamma_i} z(t), \quad t > 0. \tag{14}$$

If condition (A_0) of Theorem 3.1 is satisfied, and l is defined as in Lemma 3.3 by (12), then, if we suppose $y \in \mathscr{S}'_+$, Theorem 3.1 implies that (14) has a unique solution $z \in \mathscr{S}'_+$ given by

$$z = l * \int_0^2 \phi_1(\gamma) D^\gamma y d\gamma.$$

Equation (9) becomes

$$y(t) = -l * \int_0^2 \phi_1(\gamma) D^{\gamma-2} y(t) d\gamma + D^{-2} f(t, y(t)) + v_0 t + y_0, \quad t > 0, \tag{15}$$

so a mild solution to (CP) will exist if (15) has a solution $y \in C([0, \delta]) \cap \mathscr{S}'_+$, for some $\delta > 0$. To show such existence, we will define a mapping T by the right hand side of (15), and show existence of its fixed point in appropriate Banach space. The necessary conditions for T to have a fix point are to follow.

First, let us impose conditions on f:

(A_l) $f(t, u), (t, u) \in [0, \infty) \times \mathbb{R}$, is locally integrable and
there exist $a > 0$, $\alpha > 0$ and $h \in L^1([0, a])$, $h \geq 0$, such that
$|f(t, u)| \leq h(t)|u|^\alpha$, $u \in \mathbb{R}, t \in [0, a]$,

and the stronger one

$(A_l)'$ $f(t, u), (t, u) \in [0, \infty) \times \mathbb{R}$, is continuous $f(t, 0) = 0, t \in [0, \infty)$, and
there exist $\alpha > 0$, $a > 0$, $h \in C([0, a])$ with $h \geq 0$ and $h(0) = 0$, such that
$|f(t, u) - f(t, v)| \leq h(t)|u - v|$, $u, v \in \mathbb{R}, t \in [0, a]$.

Further, we introduce conditions on ϕ_1 and ϕ_2.

(Φ_1) ϕ_1 is continuous in $[c, d] \subset [0, 2)$, $\phi_1(\gamma) = 0$, $\gamma \notin [c, d]$;
ϕ_2 is of the form (13).

The following theorem holds.

Theorem 4.1. *Let* $y_0, v_0 \in \mathbb{R}$, $r > \max\{|y_0|, |v_0|\}$. *Let* (Φ_1), (A_0) *and* (A_l) *hold. There exists* $\delta = \delta(r) > 0$ *such that:*

a) (CP) *has a mild solution in* $[0, \delta]$.

b) *If* $y(0) = y_0 = 0$, *then* (CP) *has a mild solution* (y, z) *in* $[0, \delta]$, *such that* $y \in C^1([0, 1])$. *If* $\gamma_0 - \gamma_k > 1$ *then* (y, z) *is a non-impact solution.*

c) *If* $(A_l)'$ *holds and* $y_0 = v_0 = 0$, *then* (CP) *has a classical solution in* $[0, \delta]$.

Proof. We will give the joint proof for all parts of the theorem, underlying the differences if we assume (A_l) or $(A_l)'$.

Let l be defined by (12) as in Lemma 3.3 and define operator T by the right-hand side of (15), i.e.

$$Ty(t) := -l * \int_0^2 \phi_1(\gamma) D^{\gamma-2} y(t) d\gamma + D^{-2} f(t, y(t)) + v_0 t + y_0.$$

We will consider T acting on Banach spaces $C^i([0,a])$ endowed with the norms $\|y\|_i := \sup_{t \in [0,a]}\{|y^{(j)}(t)|; 0 \le j \le i\}$, $i = 0, 1, 2$ and, in particular, Banach spaces

$$C_0^i([0,a]) := \{y \in C^i([0,a]); y(0) = 0\}$$

and

$$C_{00}^i([0,a]) := \{y \in C^2([0,a]); y(0) = 0, y'(0) = 0\}$$

endowed with the same norm. We will show that T is a compact operator in quoted spaces and that T maps a closed ball into itself. Then we will apply the Schauder fixed point theorem and obtain a mild solution.

(i) T maps $C^i([0,a])$ into $C^i([0,a])$ for $i = 0, 1, 2$.

Let $y \in C([0,a])$, where a comes from (A_l) or $(A_l)'$. Define

$$Jy(t) := \int_c^d \phi_1(\gamma) D^{\gamma-2} y(t) d\gamma$$

$$= \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2-\gamma)} \int_0^t (t-x)^{1-\gamma} y(x) dx d\gamma, \quad t \in [0,a]. \tag{16}$$

Since $y \in C([0,a])$ and $\operatorname{supp} \phi_1 \subset [c,d] \subset [0,2)$, Jy is continuous. Further, denote

$$Ry(t) := (l * Jy)(t) = \int_0^t l(x) Jy(t-x) dx, \quad t \in [0,a],$$

where l is defined as in Lemma 3.3 and therefore is locally integrable. It follows that Ry is continuous. Function Gy defined by

$$Gy(t) := D^{-2} f(t, y(t)) = \int_0^t (t-x) f(x, y(x)) dx, \quad t \in [0,a]$$

is continuous by assumption (A_l). Since

$$Ty(t) := Ry(t) + Gy(t) + v_0 t + y_0, \quad t \in [0,a], \tag{17}$$

it follows $T : C([0,a]) \to C([0,a])$.

In what follows we shell need well known fact stated in next lemma.

Lemma 4.2. *(a) Let $l \in C(\mathbb{R})$ and $\operatorname{supp} l \subset [0, \infty)$. Then the convolution operator $l*$, defined by $y \mapsto l * y$, for $y \in L^1_{loc+}(\mathbb{R})$, maps continuously $C^1([0,a])$ into $C_0^1([0,a])$ and $C_0^2([0,a])$ into $C_{00}^2([0,a])$.*
(b) Let $l \in L^1_{loc+}(\mathbb{R})$. Then $y \mapsto l * y$, $y \in L^1_{loc+}(\mathbb{R})$ is a continuous mapping from $C([0,a])$ into $C_0([0,a])$, from $C_0^1([0,a])$ into $C_{00}^1([0,a])$ and from $C_{00}^2([0,a])$ into $C_{00}^2([0,a])$.

Let $y \in C_0^1([0, a])$. We will show that $Ty \in C_0^1([0, a])$. Partial integration in (16) and $y(0) = 0$ follows

$$\frac{d}{dt}Jy(t) = \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2 - \gamma)} \int_0^t (t - x)^{1-\gamma} y^{(1)}(x) dx d\gamma + y(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2 - \gamma)} t^{1-\gamma} d\gamma,$$

$$= Jy^{(1)}(t) + y(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2 - \gamma)} t^{1-\gamma} d\gamma,$$

$$= Jy^{(1)}(t) \quad t \in [0, a]. \tag{18}$$

Therefore $\frac{d}{dt}Jy$ is continuous and $J : C_0^1([0, a]) \to C_0^1([0, a])$. This, and the fact that l is locally integrable, by Lemma 4.2, implies that R maps $C_0^1([0, a])$ into $C_0^1([0, a])$ and

$$\frac{d}{dt}Ry(t) = l * \frac{d}{dt}Jy(t) = l * Jy^{(1)}(t), \quad t \in [0, a].$$

By (A_l) it follows that

$$\frac{d}{dt}Gy(t) = \int_0^t f(x, y(x)) dx, \quad t \in [0, a],$$

is continuous and $G : C_0^1([0, a]) \to C_0^1([0, a])$.
In total we have that

$$\frac{d}{dt}Ty(t) = l * Jy'(t) + \int_0^t f(x, y(x)) dx + v_0, \quad t \in [0, a] \tag{19}$$

is continuous, $Ty(0) = 0$, and therefore $T : C_0^1([0, a]) \to C_0^1([0, a])$.
Finally, let $y \in C_{00}^2([0, a])$. We will show that $Ty \in C_{00}^2([0, a])$. Since $y(0) = 0$ and $y'(0) = 0$

$$\frac{d^2}{dt^2}Jy(t) = Jy^{(2)}(t) + y'(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2 - \gamma)} t^{1-\gamma} d\gamma$$

$$+ y(0) \int_c^d \frac{\phi_1(\gamma)(1 - \gamma)}{\Gamma(2 - \gamma)} t^{-\gamma} d\gamma, \quad t \in [0, a]$$

is continuous. Thus,

$$\frac{d^i}{dt^i}Jy(t) = Jy^{(i)}(t), \quad t \in [0, a], \, i \in \{0, 1, 2\}, \tag{20}$$

$Jy(0) = Jy'(0) = 0$, so $J : C_{00}^2([0, a]) \to C_{00}^2([0, a])$. This, with Lemma 4.2, and the fact that l is locally integrable, implies that R maps $C_{00}^2([0, a])$ into $C_{00}^2([0, a])$ and

$$\frac{d^i}{dt^i}Ry(t) = l * \frac{d^i}{dt^i}Jy(t) = l * Jy^{(i)}(t), \quad t \in [0, a], \quad i \in \{0, 1, 2\}.$$

Assume now that $(A_l)'$ holds. Then

$$\frac{d^2}{dt^2}Gy(t) = f(t, y(t)), \quad t \in [0, a]$$

is continuous, $Gy(0) = \frac{d}{dt}Gy(0) = 0$ and therefore $G : C_{00}^2([0, a]) \to C_{00}^2([0, a])$. Thus, $T : C_{00}^2([0, a]) \to C_{00}^2([0, a])$ and

$$\frac{d^2}{dt^2}Ty(t) = l * Jy''(t) + f(t, y(t)), \quad t \in [0, a]. \tag{21}$$

(ii) *Compactness of T.*

Let us show that J is a compact operator in $C([0,a])$. Recall,

$$Jy(t) = \lim_{N\to\infty} \sum_{n=1}^{N} \phi_1(\gamma_n) D^{\gamma_n-2} y(t) \Delta\gamma_n$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\phi_1(\gamma_n)}{\Gamma(2-\gamma_n)} \int_0^t \frac{y(x)}{(t-x)^{\gamma_n-1}} dx \Delta\gamma_n, \quad t \in [0,a],$$

where γ_n are points of the interval $[c,d]$ in the definition of the Riemann sum defined for the integral. Since $\gamma_n - 2 \leq d - 2 < 0$, for all $n \in \mathbb{N}$, operators $D^{\gamma_n-2} : C^i([0,a]) \to C^i([0,a])$, $i = 0,1,2$, $n \in \{1,...,N\}$ are operators of fractional integration and therefore compact. J is a compact operator as a limit of compact operators in the operator norm topology. By the same argument, by (20), $J : C_0^1 \to C_0^1$ and $J : C_{00}^2 \to C_{00}^2$ are compact.

By Lemma 4.2, $l*$ is a continuous mapping: $C([0,a]) \to C([0,a])$, $C_0^1([0,a]) \to C_0^1([0,a])$ and $C_{00}^2([0,a]) \to C_{00}^2([0,a])$, respectively, and therefore R is a compact operator as a composition of a compact and a continuous operator on $C([0,a])$, $C_0^1([0,a])$ and $C_{00}^2([0,a])$, respectively.

Under the assumption (A_l), $G : C_0^1([0,a]) \to C_0^1([0,a])$ is compact. Let $M > 0$ and B_M be the ball in $C_0^1([0,a])$ ($B_M := \{y \in C_0^1([0,a]); \|y\|_1 \leq M\}$). The set $G[B_M] := \{G\theta; \theta \in B_M\}$ is uniformly bounded. Let us prove this. Condition (A_l) implies

$$|Gy(t)| \leq \int_0^t |t-x||f(x,y(x))|dx \leq aM^\alpha C, \quad t \in [0,a], \quad y \in B_M,$$

and

$$|\frac{d}{dt} Gy(t)| \leq \int_0^t |f(x,y(x))|dx \leq M^\alpha C, \quad t \in [0,a], \quad y \in B_M,$$

where $C = \max_{t\in[0,a]} \int_0^t h(x)dx$. It is also an equicontinuous family since

$$|Gy(t_1) - Gy(t_2)| \leq$$

$$\leq \int_0^{t_1} |(t_1-x)-(t_2-x)||f(x,y(x))|dx + \int_{t_1}^{t_2} |t_2-x||f(x,y(x))|dx$$

$$\leq M^\alpha C(t_1-t_2) + \sup_{x\in[t_1,t_2]} |t_2-x|M^\alpha C$$

$$\leq (t_1-t_2)M^\alpha C(1+a) \leq C'(t_1-t_2), \quad t_1,t_2 \in [0,a]$$

and

$$|\frac{d}{dt} Gy(t_1) - \frac{d}{dt} Gy(t_2)| \leq \int_{t_1}^{t_2} |f(x,y(x))|dx$$

$$\leq M^\alpha \int_{t_1}^{t_2} |h(x)|dx, \quad t_1,t_2 \in [0,a].$$

Since the function $t \mapsto \int_0^t |h(x)|dx$, $t \in [0,a]$ is absolutely continuous in $[a,b]$, it follows that the set $G[B_M]$ is equicontinuous. Therefore, Arzela-Ascoli theorem implies that $G[B_M]$ is relatively compact and G is, therefore, a compact operator in $C_0^1([0,a])$.

If we use the stronger condition $(A_l)'$, then G is compact in $C_{00}^2([0,a])$. We will show that G maps a bounded sequence $\{y_n\}_{n\in\mathbb{N}}$ in $C_{00}^2([0,a])$ to a relatively compact set $\{Gy_n\}_{n\in\mathbb{N}}$ in $C_{00}^2([0,a])$. If $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $C_{00}^2([0,a])$, then $\{y_n\}_{n\in\mathbb{N}}$ and $\{y_n'\}_{n\in\mathbb{N}}$ are equicontinuous and Arzela-Ascoli theorem assures that we have convergent subsequences, again denoted by $\{y_n\}_{n\in\mathbb{N}}$ and $\{y_n'\}_{n\in\mathbb{N}}$. Denote $c_n = Gy_n$, $n \in \mathbb{N}$. Then, for $i \in \{0,1\}$,

$$|\frac{d^i}{dt^i} c_n(t) - \frac{d^i}{dt^i} c_m(t)| \leq \int_0^t |t-x|^{1-i}|f(t,y_n(t)) - f(t,y_m(t))|$$

$$\leq D_i |y_n(t) - y_m(t)|^\alpha, \quad t \in [0,a], \tag{22}$$

where $D_i = \max_{t \in [0,a]} \int_0^t |t-x|^{1-i} |h(x)| dx$, $i \in \{0,1\}$, and

$$|\frac{d^2}{dt^2} c_n(t) - \frac{d^2}{dt^2} c_m(t)| \leq |f(t, y_n(t)) - f(t, y_m(t))|$$
$$\leq D_2 |y_n(t) - y_m(t)|^\alpha, \quad t \in [0,a], \tag{23}$$

where $D_2 = \max_{t \in [0,a]} |h(x)| dx$. Since $\{y_n\}_{n \in \mathbb{N}}$ converges, (22) and (23) imply that the same is true for $\{c_n\}_{n \in \mathbb{N}}$ in $C^2([0,a])$.

As a sum of compact operators R, G and the mapping: $y \mapsto v_0 t + y_0$, T is compact and $T : C[0,a] \rightarrow C[0,a]$. Moreover, $T : C_0^1([0,a]) \rightarrow C_0^1([0,a])$ is compact if (A_l) holds, while $T : C_{00}^2([0,a]) \rightarrow C_{00}^2([0,a])$ is compact if $(A_l)'$ holds.

(iii) *Determination of δ.*

We will show that there exists $\delta > 0$ depending on r such that T maps $B_r \subset C([0,\delta])$, $B_r \subset C_0^1([0,\delta])$ and $B_r \subset C_{00}^2([0,\delta])$ into itself.
From (16) and (20) we derive

$$|\frac{d^i}{dt^i} Jy(t)| \leq M_t \sup_{x \in [0,t]} |y^{(i)}(x)|, \quad i \in \{0,1,2\}, \tag{24}$$

where

$$M_t = \int_c^d \frac{|\phi_1(\gamma)|}{\Gamma(3-\gamma)} t^{2-\gamma} d\gamma \quad t \in [0,\delta].$$

Now consider $B_r \subset C([0,\delta])$ and $B_r \subset C_0^1([0,\delta])$, respectively. Then (17), (19), (24) and (A_l) imply that for $i = 0,1$,

$$|\frac{d^i}{dt^i} Ty(t)| \leq |l * \frac{d^i}{dt^i} Jy(t)| + |\frac{d^i}{dt^i} Gy(t)| + |v_0| t^{1-i} + |y_0|(1-i)$$
$$\leq \int_0^t |l(t-x)| |Jy^{(i)}(x)| dx + \int_0^t |(t-x)^{1-i}| |f(x, y(x))| dx + |v_0| t^{1-i} + |y_0|(1-i)$$
$$\leq D_t M_t \sup |y^{(i)}(t)| + C_t \sup |y(t)|^\alpha + |v_0| t^{1-i} + |y_0|(1-i), \quad t \in [0,\delta], \tag{25}$$

where

$$D_t = \int_0^t |l(x)| dx, \quad C_t = \int_0^t h(x)(t-x)^{1-i} dx, \quad t \in [0,\delta].$$

Since $|y^{(i)}(t)| \leq r$, $i = 0,1$ and $r > \max\{|v_0|, |y_0|\}$, it is possible to shrink δ as much as we need to reach

$$|\frac{d^i}{dt^i} Ty(t)| \leq r, \quad \text{for } i = 0,1.$$

With this, we determine δ in cases a) and b).
For the case c), consider the ball $B_r \subset C_{00}^2([0,\delta])$. Then (21), (24) and $(A_l)'$ imply that the estimates for $\frac{d^i}{dt^i} Ty(t)$, $i = 0,1$ are valid and

$$|\frac{d^2}{dt^2} Ty(t)| \leq |l * \frac{d^i}{dt^i} Jy(t)| + |f(t, y(t))|$$
$$\leq D_t M_t \sup |y^{(2)}(t)| + C_t |y(t)|^\alpha, \quad t \in [0,\delta],$$

with $C_t = \max_{t \in [0,\delta]} |h(t)|$ and D_t and M_t as above. Since $|y^{(i)}(t)| \leq r$, $i = 0,1,2$ and $r > 0$, it is possible to shrink δ such that

$$|\frac{d^i}{dt^i} Ty(t)| \leq r, \quad \text{for } i = 0,1,2,$$

so it determines δ.

(iv) *Assertions a), b) and c).*

a) Recall that assuming (A_l) and fixing $r > \max\{|y_0|, |v_0|\}$, by (i)-(iv), we have obtained the existence of $\delta = \delta(r) > 0$, such that T is a compact mapping $C([0, \delta]) \to C([0, \delta])$ and that T maps $B_r \subset C([0, \delta])$ into itself. Thus, according to the Schauder fixed point theorem, T has a fixed point in $B_r \subset C([0, \delta])$, i.e. (15) has a solution $y \in B_r \subset C([0, \delta])$. Set

$$z = l * \int_c^d \phi_1(\gamma) D^\gamma y \, d\gamma, \quad \text{with } l = \mathcal{L}^{-1}\left(\frac{1}{\sum_{i=0}^k a_i s^{\gamma_i}}\right). \tag{26}$$

Then (y, z) is a mild solution to (CP).

b) Again, assuming (A_l) and fixing r, by (i)-(iv), we find $\delta = \delta(r) > 0$, such that $T : C_0^1([0, \delta]) \to C_0^1([0, \delta])$ is compact and T maps the ball $B_r \subset C_0^1([0, \delta])$ into itself. Again, the use of the Schauder fixed point theorem assures that T has a fixed point y in $B_r \subset C_0^1([0, \delta])$. If z is given by (26) then (y, z) is a mild solution to (CP) in $[0, \delta]$.

If $\gamma_0 - \gamma_k > 1$, then Lemma 3.3 implies that l is absolutely continuous. Thus, $l * Jy \in AC^2([0, a])$. Also $Gy \in AC^2([0, \delta])$. By (17) and the fact that $y = Ty$ we obtain $y \in AC^2([0, a])$.

c) With $(A_l)'$ and $r > 0$, by (i)-(iv), we find $\delta = \delta(r) > 0$ such that T is compact and maps $B_r \subset C_{00}^2([0, a])$ into itself. According to the Schauder theorem, (15) has a solution y in $C_{00}^2([0, \delta])$. If z is given by (26) then (y, z) is a mild solution to (CP) in $[0, \delta]$. $D^\gamma y$, $\gamma < 2$ is continuous, since $y \in C_{00}^2$, and $z \in C([0, \delta])$ by (26). Applying D^2 to y given by (15) leads to conclusion that (y, z) satisfies $\widetilde{(CP)}_1$.

$\square$

Different conditions on ϕ_1 and f

Further, we assume restriction on $\operatorname{supp} \phi_1$, which will imply better regularity properties of solutions. Let

$$(\Phi_2) \quad \phi_1 \text{ is continuous in } [c, d] \subset [0, 1), \phi_1(\gamma) = 0, \gamma \notin [c, d];$$
$$\phi_2 \text{ is of the form (13).}$$

Then, we have the following result:

Theorem 4.3. *Let $y_0, v_0 \in \mathbb{R}$ and $r > \max\{|y_0|, |v_0|\}$. Let (Φ_2), (A_0) and (A_l) hold. Then there exists $\delta = \delta(r) > 0$, such that:*

a) *(CP) has a non-impact solution in $[0, \delta]$.*

b) *If $(A_l)'$ holds and $y_0 = 0$, then (CP) has a classical solution in $[0, \delta]$.*

Proof. We consider operator T defined by (17) and proceed in the same way as in Theorem 4.1. The proof of this theorem is very similar to the proof of Theorem 4.1, therefore we will just give the parts of the proof which are different.

(i) *T maps $C^i([0, a])$ into $C^i([0, a])$ for $i = 1, 2$.*

Let $y \in C^1([0, a])$. The continuity of Ty follows as in Theorem 4.1. Let us show the continuity of its first derivative $\frac{d}{dt} Ty$.

An expression for operator $\frac{d}{dt} J$ (see (18)) has an additional summand, which does not vanish (we do not have $y(0) = 0$), but continuity follows from the fact that $\phi_1(\gamma) = 0$ for $\gamma \geq 1$. Therefore,

$$\frac{d}{dt} Ty(t) = \frac{d}{dt}(l * Jy)(t) + Gy(t) + v_0, \tag{27}$$

$$= l * Jy'(t) + l * y(0) \int_0^d \frac{\phi_1(\gamma)}{\Gamma(2 - \gamma)} t^{1-\gamma} d\gamma + \int_0^t f(t, y(t)) + v_0, \quad t \in [0, a],$$

and thus $T : C^1([0,a]) \to C^1([0,a])$.

Let $y \in C_0^2([0,a])$ and assume $(A_l)'$ instead of (A_l). Then, $Ty(0) = 0$ because $y(0) = 0$ and the first derivative of Ty is given by (19). The second derivative given by

$$\frac{d^2}{dt^2} Ty(t) = l * Jy''(t) + l * y'(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2 - \gamma)} t^{1-\gamma} d\gamma + f(t, y(t)), \quad t \in [0,a]$$

is continuous since $y'' \in C([0,a])$, $J : C([0,a]) \to C([0,a])$, $\phi_1 \equiv 0$ for $\gamma \geq 1$ and f is continuous. Therefore $T : C_0^2([0,a]) \to C_0^2([0,a])$.

(ii) Additional regularity properties of T.

Additional properties of T which assure higher regularity of the solutions are to follow. We will show that (A_l) implies the mapping properties

$$T : C([0,a]) \to C^1([0,a]) \quad \text{and} \quad T : C^1([0,a]) \to AC^2([0,a]). \tag{28}$$

We will use the following lemma.

Lemma 4.4. *a) Let $y \in C([0,a])$ and $\alpha \geq 1$. Then $I^\alpha y \in C^1([0,a])$.*
b) Let $y \in L^1_{\text{loc}}([0,a])$ and $\alpha \geq 1$. Then $I^\alpha y \in AC([0,a])$.

Proof. a) If $\alpha = 1$ then $\frac{d}{dt} I^\alpha y(t) = y(t)$ and therefore if $y \in C([0,a])$ then $I^\alpha y \in C^1([0,a])$, and if $y \in L^1_{\text{loc}}([0,a])$ then $I^\alpha y \in AC([0,a])$. For $\alpha > 1$ using that $I^\alpha I^\beta = I^{\alpha+\beta}$ for $\alpha, \beta > 0$ and $D^1 I^1 = I$ we have that $\frac{d}{dt} I^\alpha y(t) = D^1 I^1 I^{\alpha-1} y(t) = I^{\alpha-1} y(t)$. If $y \in C([0,a])$ then $I^{\alpha-1} y \in C([0,a])$, and again $I^\alpha y \in C^1([0,a])$. If $y \in L^1_{\text{loc}}([0,a])$ then $I^{\alpha-1} y \in L^1_{\text{loc}}([0,a])$ (since $I^{\alpha-1} : L^1_{\text{loc}}([0,a]) \to L^1_{\text{loc}}([0,a])$) and therefore $I^\alpha y \in AC([0,a])$. $\qquad \square$

Let $y \in C([0,a])$. Lemma 4.4 gives $I^\alpha y \in C^1([0,a])$ if $\alpha > 1$, hence

$$J : C([0,a]) \to C^1([0,a]). \tag{29}$$

Since $Jy(t) = \int_c^d \phi_1(\gamma) D^{\gamma-2} y \, d\gamma, t \in [0,a]$ and $d < 1$, it follows that operator $D^{\gamma-2}$ is in fact $I^{2-\gamma}$ with $2 - \gamma > 1$; so $D^{\gamma-2} y \in C^1([0,a])$. Since $Gy \in C^1([0,a])$ we have that $Ty \in C^1([0,a])$.

Let $y \in C^1([0,a])$. First note that if $\gamma < 1$ then $(t^{1-\gamma})' = (1-\gamma) t^{-\gamma} \in L^1_{\text{loc}}(\mathbb{R})$, hence

$$y(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2-\gamma)} t^{1-\gamma} d\gamma \in AC([0,a]). \tag{30}$$

Assumption $y' \in C([0,a])$, as well as (29) and (30) imply that

$$\frac{d}{dt} Jy = Jy' + y(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2-\gamma)} t^{1-\gamma} d\gamma \in AC([0,a]).$$

Thus, $J : C^1([0,a]) \to AC^2([0,a])$. Further, $Gy \in AC^2([0,a])$ because $\frac{d^2}{dt^2} Gy(t) = f(t, y(t))$, $t \in [0,a]$ and for y continuous $t \mapsto f(t, y(t))$ is a locally integrable function. Therefore, by (27) we have that $T : C^1([0,a]) \to AC^2([0,a])$.

(iii) T is compact in $C^i([0,a])$, $i = 1, 2$.

Let $i = 1$. By Theorem 4.1, T, given by (17) (with (19)), maps a bounded sequence in $C^1([0,a])$ into a sequence with a convergent subsequence in $C^1([0,a])$.

For the compactness of the operator T, given by (17) (with (27)), we need additionally to show that if $\{y_n\}_{n \in \mathbb{N}}$ is bounded in $C^1([0,a])$, then

$$\left\{ y_n(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2-\gamma)} l(t) * t^{1-\gamma} d\gamma \right\}_{n \in \mathbb{N}}$$

has a convergent subsequence. This is true since any bounded sequence $\{y_n\}_{n\in\mathbb{N}}$ in $C^1([0,a])$ has a subsequence, again denoted by $\{y_n\}_{n\in\mathbb{N}}$, which is convergent in $C([0,a])$. Hence, $\{y_n(0)\}_{n\in\mathbb{N}}$ converges as well.

Similarly, one can prove the compactness of T in $C_0^2([0,a])$.

(iv) Determination of δ.

Take $B_r \subset C^1([0,\delta])$ and $B_r \subset C_0^2([0,\delta])$, respectively, and δ will be defined later. Then, for $i \in \{0,1\}$,

$$|l * y^{(i)}(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2-\gamma)} t^{1-\gamma} d\gamma| \le |y^{(i)}(0)| M_t' \le M_t' \|y\|_{i+1}, \quad t \in [0,\delta],$$

where

$$M_t' = \int_0^t l(\tau) d\tau \int_c^d \frac{|\phi_1(\gamma)|}{|\Gamma(2-\gamma)|} t^{1-\gamma} d\gamma, \quad t \in [0,\delta].$$

Using this, (27) and estimate (25), similarly as in the proof of Theorem 4.1, one can estimate $\frac{d^i}{dt^i} Ty(t)$, for $i = 0,1$, if (A_l) is assumed, and for $i = 0,1,2$, if $(A_l)'$ is assumed. Therefore, it is possible to find $\delta > 0$ such that $y \in B_r \subset C^1([0,\delta])$ and $y \in B_r \subset C_0^2([0,\delta])$, respectively, implies that

$$|\frac{d^i}{dt^i} Ty(t)| \le r, \quad i = 0,1, \text{ respectively } i = 0,1,2.$$

(v) Assertions a) and b).

a) The Schauder fixed point theorem implies the existence of a fixed point y for T in $C^1([0,\delta])$, if (A_l) is assumed. Let z be given by (26). Then (y,z) is mild solution to (CP). By (28) we obtain that $y \in AC^2([0,\delta])$. This, with $d < 1$, implies

$$\int_c^d \phi_1(\gamma) D^\gamma y(t) d\gamma \in AC([0,\delta]).$$

Since l is locally integrable, we have that $z \in AC([0,\delta])$. Therefore, the mild solution (y,z) satisfies the first equation in (CP) in $L_{\text{loc}}^1([0,\delta])$, so this is also a non-impact solution

b) If we assume $(A_l)'$, then T has a fixed point in $C_0^2([0,\delta])$, (CP) has a mild solution (y,z) in $[0,\delta]$, z is given by (26), and $y \in C_0^2([0,\delta])$. So $\widetilde{(CP)}_2$ holds. Further, $d < 1$ implies that $\int_c^d \phi_1(\gamma) D^\gamma y(t) d\gamma \in C^1([0,\delta])$ and therefore, $z \in C^1([0,\delta])$ and $\widetilde{(CP)}_1$ is satisfied for all $t \in [0,\delta]$. $\square$

In this particular case, one can also impose a condition on f, stronger then (A_l) and $(A_l)'$, and obtain classical solution to (CP) in $[0,\delta]$, which satisfies $\widetilde{(CP)}_2$ for all $t \in [0,\delta]$. The condition reads:

$(A_l)''$ $f(t,u)$, $(t,u) \in [0,\infty) \times \mathbb{R}$, is continuous $f(t,0) = 0, t \in [0,\infty)$, and
     there exist $a > 0$, $h \in C([0,a])$ with $h \ge 0$ and $h(0,0) = 0$, such that
     $|f(t,u) - f(s,v)| \le h(t,s)(|t-s| + |u-v|)$, $u,v \in \mathbb{R}, t \in [0,a]$.

The corresponding theorem is the following.

Theorem 4.5. *Let $v_0 \in \mathbb{R}$, $r > |v_0|$ and $y_0 = 0$. Let (Φ_2), (A_0) and $(A_l)''$ hold. Then there exists $\delta = \delta(r) > 0$, such that there exists the classical solution to (CP) in $[0,\delta]$, which satisfies equation $(CP)_2$ point-wisely for $t \in [0,\delta]$.*

Proof. First note that all assumptions of Theorem 4.3 are satisfied. Hence there is a classical solution to (CP) in $[0,\delta]$ obtained in five steps in previous proof. The assumption $(A_l)''$ gives

stronger results. Note that if $y \in AC([0,a])$ and $(A_l)''$ holds, then $t \mapsto f(t, y(t))$, $t \in [0,a]$ is absolutely continuous. Indeed,

$$|f(t, y(t)) - f(s, y(s))| \leq h(t,s)(|t-s| + |y(t) - y(s)|)$$

implies absolutely continuity since y belongs to $AC([0,a])$.

Further note that in step *(ii)* of the proof of the Theorem 4.3, in addition to (28) we have

$$T : C^2([0,a]) \to AC^3([0,a]). \tag{31}$$

To see this, let $y \in C_0^2([0,a])$. Then (29) and (30) imply that

$$\frac{d^2}{dt^2} Jy = Jy'' + y'(0) \int_c^d \frac{\phi_1(\gamma)}{\Gamma(2-\gamma)} t^{1-\gamma} \in AC^3([0,a]).$$

Since $t \mapsto f(t, y(t))$, $t \in [0,a]$ is absolutely continuous, we have that $Gy \in AC^3([0,a])$ and therefore (31). It follows that the fixed point $y \in C_0^2([0,a])$ obtained in part b) of the fifth step of the proof of the Theorem 4.3 is then an element of $AC^3([0,\delta])$. Then z given by (26) is an element in $AC^2([0,\delta])$, which yields that both equations in (CP) are satisfied for all $t \in [0,\delta]$. $\qquad\square$

Remark 4.6. a) Note that in Theorem 4.1 and Theorem 4.3 we could have assumed that $[c,d]$ was a subset of the interval $(-\infty, 2)$ and $(-\infty, 1)$, respectively, in which case we would have the same results. Indeed, with respective assumptions, we change the bounds of the integral in (16), the definition of operator J, but not its properties: $\frac{d^i}{dt^i} Jy$, $i = 0, 1, 2$ remain continuous, J remains compact in respective spaces, and estimates in (25) hold.

b) With assumptions $d < 2$ and $d < 1$, respectively we could also have considered a constitutive equation of the form

$$\sum_{i=0}^k a_i D^{\gamma_i} z(t) = \int_{-\infty}^d \phi_1(\gamma) D^\gamma y(t) d\gamma$$

and we would obtained the same results as in Theorem 4.1 and in Theorem 4.3, respectively.

c) It also make a sense to consider constitutive equation for $d < 0$. Then on its right hand side only fractional integrals of y appears. Such case is covered with condition that follows.

(Φ_3) ϕ_1 is continuous function in $[c,d] \subset (-\infty, 0)$, $\phi_1(\gamma) = 0$, $\gamma \notin [c,d]$;

   ϕ_2 is of the form (13).

Also, the theorem similar to Theorems 4.1 and 4.3 holds.

Theorem 4.7. *Let $y_0, v_0 \in \mathbb{R}$ and $r > \max\{|y_0|, |v_0|\}$. Let (Φ_3), (A_0) and $(A_l)'$ hold. Then there exists $\delta = \delta(r) > 0$ such that (CP) has a classical solution in $[0,\delta]$. Moreover, the classical solution satisfies $(CP)_2$ for all $t \in [0,\delta]$.*

Finally, we are interested in cases when both ϕ_1 and ϕ_2 are linear combinations of translations of delta distributions:

(Φ_4) $\phi_1(\cdot) = \sum_{j=0}^m b_j \delta(\cdot - \beta_j), \beta_i \in [0,2), \beta_0 \geq \beta_j \geq \beta_m, j \in \{0, ..., m\}$

   ϕ_2 is of the form (13).

The constitutive equation $(CP)_2$ becomes

$$\sum_{i=0}^k a_i D^{\gamma_i} z(t) = \sum_{j=0}^m b_j D^{\beta_j} y(t), \quad t > 0,$$

and the theorem similar to previous holds.

Theorem 4.8. *Let $y_0, v_0 \in \mathbb{R}$ and $r > \max\{|y_0|, |v_0|\}$. Let (Φ_4), (A_0) and (A_l) hold. Then there exists $\delta = \delta(r) > 0$ such that:*

a) *(CP) has a mild solution in $[0, \delta]$.*

b) *(CP) has a non-impact solution in $[0, \delta]$ if one of the following conditions hold*

$$(i)\, y_0 = 0 \text{ and } \gamma_0 - \gamma_k > 1 \quad or \quad (ii)\, \beta_0 < 1.$$

c) *If $(A_l)'$ holds, then (CP) has a classical solution in $[0, \delta]$ if one of the following conditions hold*

$$(i)\, y_0 = v_0 = 0 \quad or \quad (ii)\, \beta_0 < 1 \text{ and } y_0 = 0 \quad or \quad (iii)\, \beta_0 < 0.$$

The proof similar to previous ones is skipped.

5 Continuous ϕ_2

In this section we impose different condition to ϕ_2.

$(\Phi_5):$ • ϕ_1 is continuous function in $[c, d] \subset [0, 2)$, $\phi_1(\gamma) = 0$, $\gamma \notin [c, d]$ or

$$\phi_1(\cdot) = \sum_{j=0}^{m} b_j \delta(\cdot - \beta_j), \beta_i \in [0, 2), \beta_0 \geq \beta_j \geq \beta_m, j \in \{0, ..., m\}$$

• $\phi_2 \in C^3([0, 1])$, $\phi_2 \equiv 0$ out of $[0, 1]$, $\phi_2(1) \neq 0$ and
either $\phi(0) \neq 0$ or $\phi_2(\gamma) \sim p\gamma^q$, $p > 0$, $q > 0$.

Assumptions on ϕ_2 given in (Φ_5) will imply (as it is shown in [18]) the existence of a solution to distributed order differential equation $\int_0^1 \phi_2(\gamma) D^\gamma y = g$, i.e. the existence of a locally integrable function $\mathcal{L}^{-1}(\frac{1}{\int_0^1 \phi_2(\gamma) s^\gamma d\gamma})$.

Theorem 5.1. *Let $y_0, v_0 \in \mathbb{R}$, $r > \max\{|y_0|, |v_0|\}$ and f satisfy (A_l). Let ϕ_1 and ϕ_2 satisfy (Φ_5). Then there exists $\delta = \delta(r) > 0$ such that:*
a) There exists a mild solution of (CP) in $[0, \delta]$.
b) There exists a non-impact solution of (CP) in $[0, \delta]$ if $\operatorname{supp} \phi_1 \subset [c, 1)$.
c) There exists a classical solution of (CP) in $[0, \delta]$ if $(A_l)'$ and one of the following conditions hold:
(i) $y_0 = v_0 = 0$ or (ii) $\operatorname{supp} \phi_1 \subset [c, 1)$ and $y_0 = 0$ or (iii) $\operatorname{supp} \phi_1 \subset [c, 0)$.

Proof. The solution to equation

$$\int_0^1 \phi_2(\gamma) D^\gamma z d\gamma = g, \quad \text{in } \mathscr{S}'_+$$

is given by

$$z = \chi * g, \text{ where } \chi = \mathcal{L}^{-1}\left(\frac{1}{\int_0^1 \phi_2(\gamma) s^\gamma d\gamma}\right),$$

provided that the inversion exist. It is proved in [18, Proposition 3.1], that conditions on ϕ_2 given in (Φ_5), imply that χ exists and, moreover, that $\chi \in C^\infty((0, \infty)) \cap L^1_{\text{loc}}([0, \infty])$. Therefore, $(CP)_2$

$$\int_0^1 \phi_2(\gamma) D^\gamma z d\gamma = \int_0^2 \phi_1(\gamma) D^\gamma y d\gamma$$

has a solution in $\mathscr{S}'_+$

$$z = \chi * \int_0^2 \phi_1(\gamma) D^\gamma y d\gamma.$$

The substitution of z in $(CP)_1$ and the integrations give

$$y = -\chi * \int_0^2 \phi(\gamma) D^{\gamma-2} y d\gamma + D^{-2} f(\cdot, y(\cdot)) + v_0 t + y_0, \quad \text{in } \mathscr{S}'_+.$$

Again, we consider operator T which is given by (17), where R is now given by $Ry := \chi * Jy$ and J by (16). Since χ is locally integrable, the same properties hold for J, R and T, as in Theorems 4.1 and others. Thus, it follows that there exist $\delta = \delta(r)$ and a mild solution (y, z) to (CP) in $[0, \delta]$. Also, one can prove the properties of y and z quoted in a), b), and c) in the same way as in Theorems 4.1 and others.

$\square$

To close the section we give the example that follows arise from application. Consider the system (CP) with $\phi_1(\gamma) = b^\gamma$ and $\phi_2(\gamma) = a^\gamma$, $\gamma \in [0, 2)$, where a and b are positive constants with $b > a$. The latter condition is consequence of the Second Law of thermodynamics.

$$(CP)_{ex} \quad \begin{cases} D^2 y(t) + z(t) = f(t, y(t)), \quad t > 0, \\[2mm] \int_0^1 b^\gamma D^\gamma y(t) d\gamma = \int_0^1 a^\gamma D^\gamma z(t) d\gamma, \quad t > 0 \\[2mm] y(0) = y_0; \qquad y^{(1)}(0) = v_0, \end{cases}$$

One can apply Theorem 5.1 to prove the existence of a mild and a classical solution for above system.

Theorem 5.2. *Let $y_0, v_0 \in \mathbb{R}$ and $r > \max\{|y_0|, |v_0|\}$. Let (A_l) hold. Then problem $(CP)_{ex}$*
(i) has a non-impact solution.
(ii) has a classical solution (y, z) if $(A_l)'$ holds and if $y_0 = 0$.

Proof. Since $\phi_1, \phi_2 \in C^3([0, 1])$, $\phi_2(1) = b \neq 0$ and $\phi_2(0) = 1 \neq 0$ conditions of Theorem 5.1 (part b) case (ii) and part c) case (ii)) are satisfied. Theorem 5.1 implies the assertions. $\square$

References

[1] T. M. Atanackovic. A generalized model for the uniaxial isothermal deformation of a viscoelastic body. *Acta Mechanica*, 30:1–10, 2002.

[2] T. M. Atanackovic. On a distributed derivative model of viscoelastic body. *CRAS Mechanics*, 331:687–692, 2003.

[3] T. M. Atanackovic, M. Budimčević, and S. Pilipović. On a fractional distributed-order oscilator. *J. Phys. A: Math.Gen*, 38:6703–6713, 2005.

[4] T. M. Atanackovic, Lj. Oparnica, and S. Pilipović. On a nonlinear distributed order fractional differential equation. *J. Math. Anal. Appl.*, 328:590–608, 2007.

[5] T. M. Atanackovic, Lj. Oparnica, and S. Pilipović. Distributional framework for solving fractional differential equation. *Int. Transf. Spec. Func.*, 20:215–222, 2009. available at: arXiv:0902.0496.

[6] T. M. Atanackovic and S. Pilipović. On a class of equations arising in linear viscoelasticity theory. *ZAMM*, 85, 2005.

[7] R. L. Bagely and P. J. Torvik. On the existence of the order domain and the solution of distributed order equations i and ii. *Int. J. Appl. Math 2*, pages 865–882 and 965–987, 2000.

[8] R. L. Bagley and P. J. Torvik. On the appearance of the fractional derivative in the behavior of real materials. *J. of Appl. Mech.*, pages 294–298, 1984.

[9] M. Caputo. Linear models of disipation whose q is almost frequency independent. *Geophys.J.R.astr.Soc.*, 13:529–539, 1967.

[10] M. Caputo. Distributed order differential equation modeling dielectric induction and diffusion. *Fractional Calculus and Applied Analysis 4*, pages 421–442, 2001.

[11] M. Caputo and F. Mainardi. A new dissipation model based on memory mechanism. *Pure and Applied Geophysics*, 91:134–147, 1971.

[12] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov. Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equation. *Phys. Rev. E 66*, 046129:1–6, 2002.

[13] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, and V. Y. Gonchar. Distributed order time fractional diffusion equation. *Fractional Calculus and Applied Analysis*, 6:259–279, 2003.

[14] R. Christensen. *Theory of Viscoelasticity*. Academic Press, New York, 1982.

[15] T. T. Hartley and C. F. Lorenzo. Fractional-order system identification based on continuous order-distributions. *Appl. Analysis*, 86:1347–1363, 2007.

[16] S. Kempfle, I. Schäfer, and H. Beyer. Fractional calculus via Functional Calculus: Theory and Applications. *Nonlinear Dynamics*, 29:99–127.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[18] A. N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.*, 340:252–281.

[19] F. Mainardi and G. Pagnini. The role of the fox-wright function in fractional sub-diffusion of distributed order. *J. Comp. and Appl. Math.*, 207:245–257.

[20] F. Mainardi, G. Pagnini, and R. Gorenflo. Some aspects of fractional diffusion equation of single and distributed order. *Appl. Math. and Computation*, 187:295–305.

[21] A. M. Nakhushev. *Fractional Calculus and its Applications*. Fizmatlit, Moscow, 2003.

[22] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach Science Publishers, Amsterdam, 1993.

[23] C. Truesdell and W. Noll. *The Non-Linear Field Theories of the mechanics*. Handbuch der Physik, Band III/3, pp. 1–602 Springer-Verlag, Berlin, 1965.

[24] V. S. Vladimirov. *Generalized Function in Mathematical Phisics*. Mir, Moscow, 1979.

[25] V. S. Vladimirov, Yu. N. Drozinov, and B. I. Zavlyalov. *Tauberian Theorems for Generalized Functions*. Nauka, Moscow, 1987.